On balanced 4-holes in bichromatic point sets [*]

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Abstract

Let $S = R \cup B$ be a point set in the plane in general position such that each of its elements is colored either red or blue, where R and B denote the points colored red and the points colored blue, respectively. A quadrilateral with vertices in S is called a 4-hole if its interior is empty of elements of S. We say that a 4-hole of S is balanced if it has 2 red and 2 blue points of S as vertices. In this paper, we prove that if R and B contain n points each then S has at least $\frac{n^2-4n}{12}$ balanced 4-holes, and this bound is tight up to a constant factor. Since there are two-colored point sets with no balanced *convex* 4-holes, we further provide a characterization of the two-colored point sets having this type of 4-holes.

1 Introduction

Let S be a set of points in the plane in general position. A *hole* of S is a simple polygon Q with vertices in S and with no element of S in its interior. If Q has

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k vertices, it is called a *k-hole* of P. Note that we allow for a k-hole to be non-convex. We will refer to a hole that is not necessarily convex as *general hole*, and to a hole that is convex as simply *convex hole*. The study of *convex* k-holes in point sets has been an active area of research since Erdős and Szekeres [5, 6] asked about the existence of k points in convex position in planar point sets. It is known that any point set with at least ten points contains convex 5-holes [9]. Horton [10] proved that for $k \geq 7$ there are point sets containing no convex k-holes. The question of the existence of convex 6-holes remained open for many years, but recently Nicolás [14] proved that any point set with sufficiently many points contains a convex 6-hole. A second proof of this result was subsequently given by Gerken [8].

Recently, the study of general holes of colored point sets has been started [1, 2]. Let $S = R \cup B$ be a finite set of points in general position in the plane. The elements of R and B will be called, respectively, the *red* and *blue* elements of S, and S will be called a *bicolored* point set. A 4-hole of S is *balanced* if it has two blue and two red vertices.

In this paper, we address the following question: Is it true that any bicolored point set with at least two red and two blue points always has a balanced 4-hole? We answer this question in the positive by showing that any bicolored point set $S = R \cup B$ with $|R| = |B| \geq 2$ always has a quadratic number of balanced 4-holes. We further characterize bicolored point sets that have balanced convex 4-holes.

The study of convex k-holes in colored point sets was introduced by Devillers et al. [4]. They obtained a bichromatic point sets with 18 points that contains no convex monochromatic 4-hole. Huemer and Seara [11] obtained a bichromatic point set with 36 points containing no monochromatic 4-holes. Later, Koshelev [12] obtained another such a point set with 46 elements. Devillers et al. [4] also proved that every 2-colored Horton set with at least 64 elements contains an empty monochromatic convex 4-hole. In the same paper the following conjecture is posed: Every sufficiently large bichromatic point set contains a monochromatic convex 4-hole. This conjecture remains open, and on the other hand Aichholzer et al [2] have proved that any bicolored point set always has a monochromatic general 4-hole. Recently, a result well related with balanced 4-holes was proved by Aichholzer et al [3]: Every two-colored linearly-separable point set $S = R \cup B$ with $|R| = |B| = n$ contains at least $\frac{1}{15}n^2 - \theta(n)$ balanced general 6-holes. In a forthcoming paper, the same authors proved the lower bound $\frac{1}{45}n^2 - \theta(n)$ on such holes in the case where R and B are not necessarily linearly-separable. One can note that a balanced 6-hole with vertices V (even if R and B are linearly separable) does not always imply a balanced 4-hole with vertices $V' \subset V$ (see, e.g., Figure 1).

Our results: For balanced general 4-holes, that is, balanced 4-holes not necessarily convex, we first show that every bicolored point set $S = R \cup B$ with $|R|, |B| \geq 2$ has at least one balanced 4-hole. We then prove that if $|R| = |B| = n$ then S has at least $\frac{n^2-4n}{12}$ balanced 4-holes (Theorem 1 of Section 2), and show

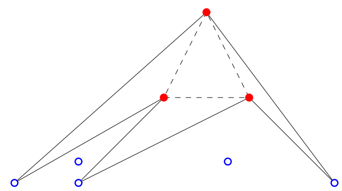


Figure 1: A balanced 6-hole such that no quadruple of its points defines a balanced 4-hole. In the whole paper, red points are represented as solid dots and blue points as tiny circles.

that this bound is tight up to a constant factor. This lower bound is improved to $\frac{2n^2+3n-8}{12}$ in the case where R and B are linearly separable (Theorem 5 of Section 2.1). On the other hand, for balanced convex 4-holes, we provide a characterization of the bicolored point sets $S = R \cup B$ having at least one such hole (Theorem 10 of Section 3.1, and Theorem 13 of Section 3.2). Finally, in Section 4, we discuss extensions of our results such as generalizing the above lower bounds for point sets in which $|R| \neq |B|$, proving the existence of convex 4-holes either balanced or monochromatic, deciding the existence of balanced convex 4-holes, and others.

General definitions: Given any two points x, y of the plane, we denote by $\overline{xy}$ the straight segment connecting x and y, by $\ell(x, y)$ the line passing through x and y, and by $x \to y$ the ray that emanates from x and contains y. For every three points x, y, z of the plane, we denote by Δxyz the open triangle with vertex set $\{x, y, z\}$. Given $X \subseteq S$, let $CH(X)$ denote the convex hull of X.

Given three non-collinear points a, b, and c, we denote by $\mathcal{W}(a, b, c)$ the open convex region bounded by the rays $a \to b$ and $a \to c$. Given a set $X \subset S$, let $f(a, b, c, X)$ denote a point $x \in (X \cap \Delta abc) \cup \{c\}$ minimizing the area of Δabx over all points of $(X \cap \Delta abc) \cup \{c\}$.

2 Lower bounds for general balanced 4-holes

It is not hard to see that if $|R|, |B| \geq 2$, then S contains a balanced 4-hole. To prove this, observe that for every set H of four points there always exists a simple polygon whose vertices are the elements of H. Let S' be a subset of S containing exactly two red points and two blue points, such that the area of the convex hull of S' is minimum. Clearly, any simple polygon whose vertex set is S' contains no element of S in its interior, and thus it is a balanced 4-hole of S.

On the other hand, if S has exactly two points of one color and many points of the other color, then S might contain only a constant number of balanced 4-holes. For example, the reader may verify that the point set of Figure 2 contains exactly five balanced 4-holes.

In the case where $|R| = |B| = n$, S has (at least) a linear number of balanced

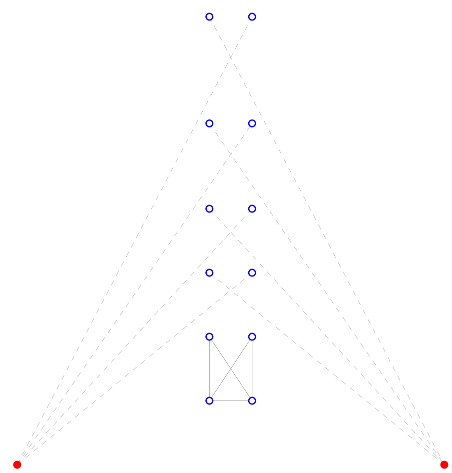


Figure 2: A point set with exactly five balanced 4-holes, obtained by choosing the two red points and any pair of blue points connected by a continuous segment.

4-holes. Indeed, by applying the ham-sandwich theorem recursively, we can partition S into a linear number of constant size disjoint subsets whose convex hulls are pairwise disjoint, and each of them contains at least two red points and two blue points, and has thus a 4-hole.

In this section we prove the following stronger result:

Theorem 1. *Let $S = R \cup B$ be a set of $2n$ points in general position in the plane such that $|R| = |B| = n$. Then S has at least $\frac{n^2-4n}{12}$ balanced 4-holes.*

We consider some definitions and preliminary results to prove Theorem 1. In the rest of this section we will assume that $|R| = |B| = n$.

Given two points $p, q \in S$ with different colors, let $T(p, q)$ be the set of the at most four points obtained by taking the first point found in each of the next four rotations: the rotation of $p \to q$ around p clockwise; the rotation of $p \to q$ around p counter-clockwise; the rotation of $q \to p$ around q clockwise; and the rotation of $q \to p$ around q counter-clockwise.

We classify (or color) the edge $\overline{pq}$ with one of the following four colors: *green*, *black*, *red*, and *blue*. We color $\overline{pq}$ green if it is an edge, or a diagonal, of some balanced 4-hole. If $\overline{pq}$ is an edge of the convex hull of S and is not green, then $\overline{pq}$ is colored black. If $\overline{pq}$ is neither green nor black, then all the points in $T(p, q)$ must have the same color and there are elements of $T(p, q)$ to each side of $\ell(p, q)$. We then color $\overline{pq}$ with the color of the points in $T(p, q)$.

Lemma 2. *The number of red edges and the number of blue edges are each at most $n\lfloor \frac{n-1}{3} \rfloor$.*

Proof. Let $r \in R$ be any red point. Sort the elements B radially around r in

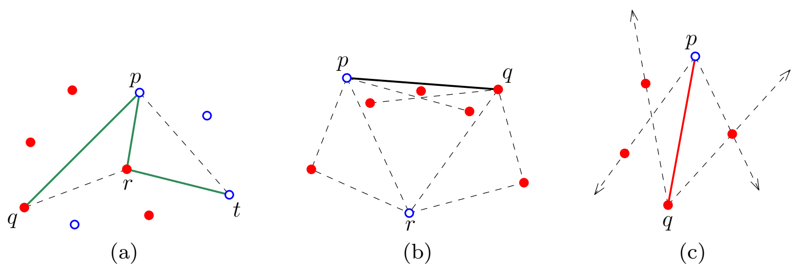


Figure 3: The edge colors: (a) The polygon with vertex set $\{p, q, r, t\}$ is a balanced 4-hole, then the edges $\overline{pq}$, $\overline{pr}$, y $\overline{rt}$ are colored green. (b) Since the edge $\overline{pq}$ is a convex hull edge and there is no balanced 4-hole with edge $\overline{pq}$, then $\overline{pq}$ is colored black. (c) Since $\overline{pq}$ is neither red nor black, and the elements of $T(p, q)$ are red, $\overline{pq}$ is colored red.

counter-clockwise order, and label them $b_0, b_1, \ldots, b_{n-1}$ in this order. Subindices are taken modulo n.

Suppose that the edge $\overline{rb_i}$ is red, $0 \leq i < n$, and the angle needed to rotate the ray $r \to b_i$ counter-clockwise around r in order to reach $r \to b_{i+1}$ is less than π. If Δrb_ib_{i+1} does not contain elements of R, then there must exist a red point z in $\mathcal{W}(rb_ib_{i+1}) \setminus \Delta rb_ib_{i+1}$. Then, the quadrilateral with vertex set $\{r, b_i, z', b_{i+1}\}$ is a balanced 4-hole, where $z' := f(b_i, b_{i+1}, z, R)$, which contradicts that $\overline{rb_i}$ is red (see Figure 4a). Hence, Δrb_ib_{i+1} must contain red points. In fact, Δrb_ib_{i+1} contains at least three red points in order to avoid that r, b_i, and b_{i+1}, joint with some red point in Δrb_ib_{i+1}, form a balanced 4-hole with edge $\overline{rb_i}$ (see Figure 4b and Figure 4c). These observations imply that the number of red edges among $\overline{rb_0}, \overline{rb_1}, \ldots, \overline{rb_{n-1}}$ (i.e. the number of red edges incident to r) is at most $\lfloor \frac{n-1}{3} \rfloor$. Summing over all the red points, the total number of red edges is at most $n\lfloor \frac{n-1}{3} \rfloor$.

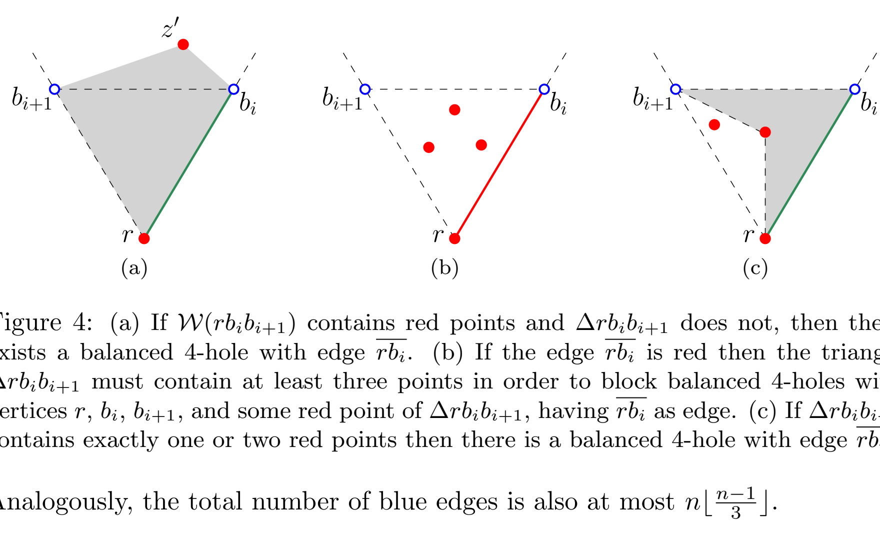


Figure 4: (a) If $\mathcal{W}(rb_ib_{i+1})$ contains red points and Δrb_ib_{i+1} does not, then there exists a balanced 4-hole with edge $\overline{rb_i}$. (b) If the edge $\overline{rb_i}$ is red then the triangle Δrb_ib_{i+1} must contain at least three points in order to block balanced 4-holes with vertices r, b_i, b_{i+1}, and some red point of Δrb_ib_{i+1}, having $\overline{rb_i}$ as edge. (c) If Δrb_ib_{i+1} contains exactly one or two red points then there is a balanced 4-hole with edge $\overline{rb_i}$.

Analogously, the total number of blue edges is also at most $n\lfloor \frac{n-1}{3} \rfloor$. $\square$

Lemma 3. *The number of green edges is at least* $\frac{n^2-4n}{3}$.

Proof. There are n^2 bichromatic edges in total. By Lemma 2, at most $n\lfloor\frac{n-1}{3}\rfloor$ of them are red and at most $n\lfloor\frac{n-1}{3}\rfloor$ are blue. Further observe that at most $2n$ edges are black. Then the number of green edges is at least:

$$n^2 - 2n\left\lfloor\frac{n-1}{3}\right\rfloor - 2n \geq \frac{n^2-4n}{3}.$$

□

Observe now that any balanced 4-hole defines at most four green edges as polygonal edges or diagonals. Thus, by Lemma 3, the number of balanced general 4-holes is at least $\frac{1}{4}\left(\frac{n^2-4n}{3}\right) = \frac{n^2-4n}{12}$, and Theorem 1 thus follows.

2.1 The separable case

We now improve our bounds of the previous section for the case where R and B are linearly separable. Suppose without loss of generality that there is a horizontal line ℓ such that the elements in R are above ℓ, and those in B are below ℓ. Further assume that no two elements in $S = R \cup B$ have the same y-coordinate.

Lemma 4. *If R and B are linearly separable then both the number of red edges and the number of blue edges are each at most* $\frac{n^2-3n+2}{6}$.

Proof. Label the red points $r_0, r_1, \ldots, r_{n-1}$ in the ascending order of the y-coordinates. Let r_i be any red point, $0 \leq i < n$. Sort the blue points radially around r_i in counter-clockwise order and label them $b_0, b_1, \ldots, b_{n-1}$ in this order. Similarly as in the proof of Lemma 2, if $\overline{r_i b_j}$ is red, $0 \leq j < n$, then among $r_0, r_i, \ldots, r_{i-1}$ the triangle $\Delta r_i b_j b_{j-1}$ contains at least three elements if $j > 0$, and the triangle $\Delta r_i b_j b_{j+1}$ contains at least three elements if $j < n-1$. Then the number of red edges incident to r_i is at most $\lfloor\frac{i}{3}\rfloor$, and over all the red points, the number of red edges is at most

$$\sum_{i=0}^{n-1}\left\lfloor\frac{i}{3}\right\rfloor$$

If $n-1 = 3k$, for some integer k, then:

$$\sum_{i=0}^{n-1}\left\lfloor\frac{i}{3}\right\rfloor = 3\left(0+1+\ldots+(k-1)\right)+k = \frac{n^2-3n+2}{6}.$$

If $n-1 = 3k+1$, then:

$$\sum_{i=0}^{n-1}\left\lfloor\frac{i}{3}\right\rfloor = 3\left(0+1+\ldots+(k-1)\right)+2k = \frac{n^2-3n+2}{6}.$$

Finally, if $n-1=3k+2$, then:

$$\sum_{i=0}^{n-1}\left\lfloor\frac{i}{3}\right\rfloor = 3\left(0+1+\ldots+k\right) = \frac{n^2-3n}{6}.$$

Therefore, we have that the number of red edges is at most $\frac{n^2-3n+2}{6}$. Analogously, there are at most $\frac{n^2-3n+2}{6}$ blue edges in total. □

Theorem 5. *If R and B are linearly separable then the number of balanced 4-holes is at least $\frac{2n^2+3n-8}{12}$.*

Proof. Since R and B are linear separable, the number of black edges is at most 2. Using Lemma 4, we can ensure that the number of green edges is at least

$$n^2-2\left(\frac{n^2-3n+2}{6}\right)-2 \quad = \quad \frac{2n^2+3n-8}{3}.$$

Then the number of balanced 4-holes is at least $\frac{2n^2+3n-8}{12} = \frac{2n^2+3n-8}{12}$. □

We observe that our lower bounds are asymptotically tight for point sets $S = R\cup B$ with $|R|=|B|=n$. For example, if R and B are far enough from each other (i.e. any line passing through two points of R does not intersect $CH(B)$, and vice versa), R is a concave chain, and B a convex chain (see Figure 5), then the number of balanced 4-holes is precisely $(n-1)\times(n-1)$; each of them convex and formed by two consecutive red points and two consecutive blue points. This point set $R\cup B$ (without the colors) was called the *double chain* [7].

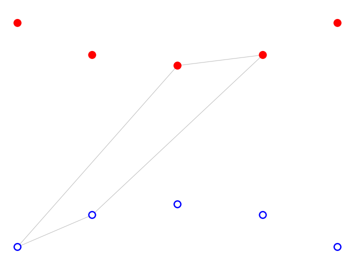


Figure 5: An example of $2n$ points having exactly $(n-1)^2$ balanced 4-holes.

3 Balanced convex 4-holes

In this section we characterize bicolored point sets $S=R\cup B$ that contain balanced *convex* 4-holes. To start with, we point out that in general $S=R\cup B$ does not have balanced convex 4-holes. The point sets shown in Figure 6 does not necessarily have balanced convex 4-holes. Observe that the number of blue points in the interior of the convex hull of the blue points in Figure 6a and

Figure 6b can be arbitrarily large. A more general example with eight points, 4 red and 4 blue linearly separable, is shown in Figure 6d, which can be generalized to point sets with $2n$ points, $n \geq 2$, n red and n blue.

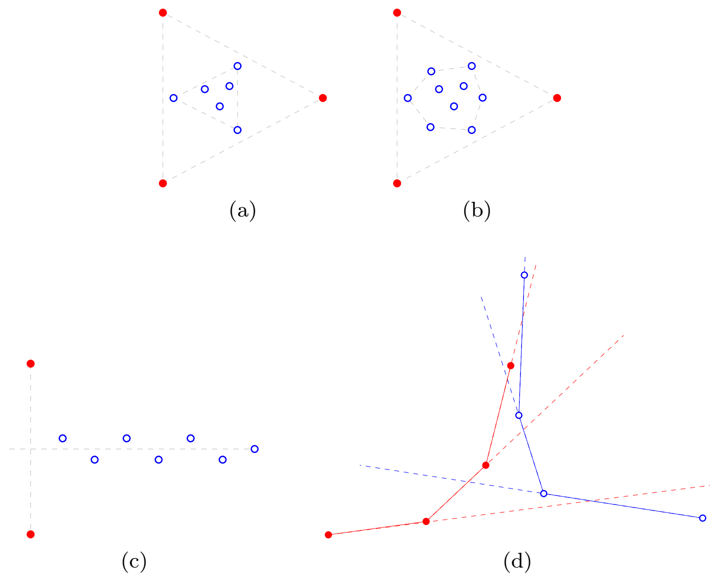


Figure 6: Some point sets with no balanced convex 4-holes.

Let $p, q \in S$ be two points of the same color. If p and q are red, $\overline{pq}$ will be called a *red-red edge*. Otherwise, if p and q are blue, we call it a *blue-blue edge*.

3.1 R and B are not linearly separable

We proceed now to characterize bicolored point sets $S = R \cup B$, not linearly separable, which contain balanced convex 4-holes. We assume $|R|, |B| \geq 2$.

Lemma 6. *If S contains a red-red edge and a blue-blue edge that intersect each other, then S contains a balanced convex 4-hole.*

Proof. Choose a red-red edge $\overline{ab}$ and a blue-blue edge $\overline{cd}$ such that $\overline{ab} \cap \overline{cd} \neq \emptyset$ and the convex quadrilateral Q with vertex set $\{a, b, c, d\}$ is of minimum area among all possible convex quadrilaterals having a red-red diagonal and a blue-blue diagonal. Observe that Q is balanced and assume that Q is not a 4-hole. Then Q contains a point of S in its interior. Suppose w.l.o.g. that there is a red point e in the interior of Q. Then we have that $\overline{ea}$ intersects $\overline{cd}$, or $\overline{eb}$ intersects $\overline{cd}$. Suppose w.l.o.g. the former case. Hence, $\{a, e, c, d\}$ is the vertex set of a balanced convex quadrilateral with a red-red diagonal and a blue-blue diagonal with area smaller than that of Q, a contradiction. $\square$

Lemma 7. *If the boundaries of $CH(R)$ and $CH(B)$ intersect each other, then S contains a balanced convex 4-hole.*

Proof. Observe that there exist a red-red edge and a blue-blue edge that intersect each other. Therefore, the result follows from Lemma 6. $\square$

Lemma 8. *Let $S = R \cup B$ be a bichromatic point set such that R and B are not linearly separable, $CH(B) \subset CH(R)$, $|R| = 3$, and $|B| \geq 2$. Then S contains a balanced convex 4-hole if and only if there is a blue-blue edge $\overline{uv}$ of $CH(B)$ such that one of the open half-planes bounded by $\ell(u, v)$ contains exactly 2 red points and no blue point.*

Proof. Let a, b, c denote the three elements of R. Suppose that there exists an edge $\overline{uv}$ of $CH(B)$ such that a and b belong to one of the two open half-planes bounded by $\ell(u, v)$ and that the elements of $S \setminus \{a, b, u, v\}$ belong to the other open half-plane (see Figure 7a). Then the quadrilateral with vertex set $\{a, b, u, v\}$ is a balanced convex 4-hole.

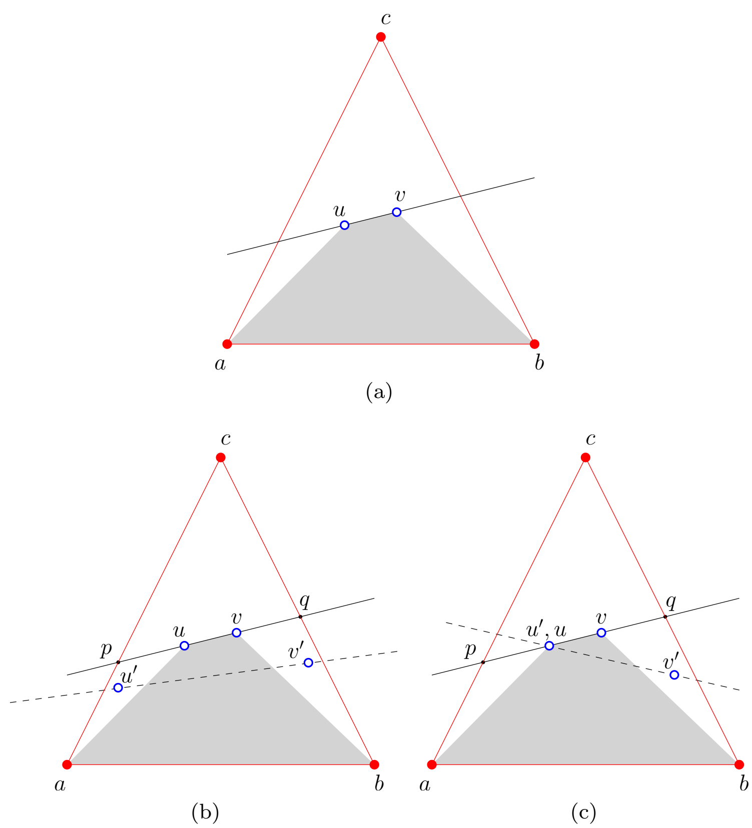


Figure 7: Proof of Lemma 8.

Suppose now that S has a balanced convex 4-hole. Assume w.l.o.g. that this 4-hole has vertex set $\{a, b, u, v\}$, where $u \to v$ intersects $\overline{bc}$, and $v \to u$ intersects ac (see Figure 7a). Let p and q denote the points $\overline{ac} \cap (v \to u)$ and $\overline{bc} \cap (u \to v)$, respectively. If $\Delta aup \cup \Delta bvq$ does not contain blue points, then $\overline{uv}$ is the edge that we are looking for. Otherwise, let blue points u' and v' be defined as follows (see Figure 7b and Figure 7c): If Δaup contains blue points then $u' := f(a, u, p, B)$, otherwise $u' := u$. Similarly, if Δbvq contains blue points then $v' := f(b, v, q, B)$, otherwise $v' := v$. Observe that the quadrilateral with vertex set $\{a, b, u', v'\}$ is a balanced convex 4-hole. Then, repeat the same arguments for u being u' and v being v'. Since at least one of the former points u and v is never considered again, and also that B is finite, after a finite number of such steps $\Delta aup \cup \Delta bvq$ will not contain blue points, and we are done. $\square$

Lemma 9. *Let $S = R \cup B$ be a bicolored point set such that R and B are not linearly separable, $CH(B) \subset CH(R)$, $|R| \geq 4$, and $|B| \geq 2$. Then S has a balanced convex 4-hole.*

Proof. Let $\mathcal{T}$ be a triangulation of R. If there are two blue points that belong to different triangles of $\mathcal{T}$, then there exist a red-red edge and a blue-blue edge intersecting each other, and the result thus follows from Lemma 6. Suppose then that B is completely contained in a single triangle t of $\mathcal{T}$, with vertices $a, b, c \in R$ in counter-clockwise order.

If $|B| = 2$, there exists and edge of $\mathcal{T}$ which is not intersected by the line through the two blue points. Then the two red points of that edge, joint with the two blue points, form a balanced convex 4-hole (Lemma 8).

Suppose then that $|B| \geq 3$, thus $CH(B)$ has at least three vertices. Since $|R| \geq 4$ there exists a triangle t' of $\mathcal{T}$ sharing an edge with t. Assume w.l.o.g. that such an edge is $\overline{ab}$, and denote by d the other vertex of t'. Further assume w.l.o.g. that $\ell(a, b)$ is horizontal, and d is below $\ell(a, b)$.

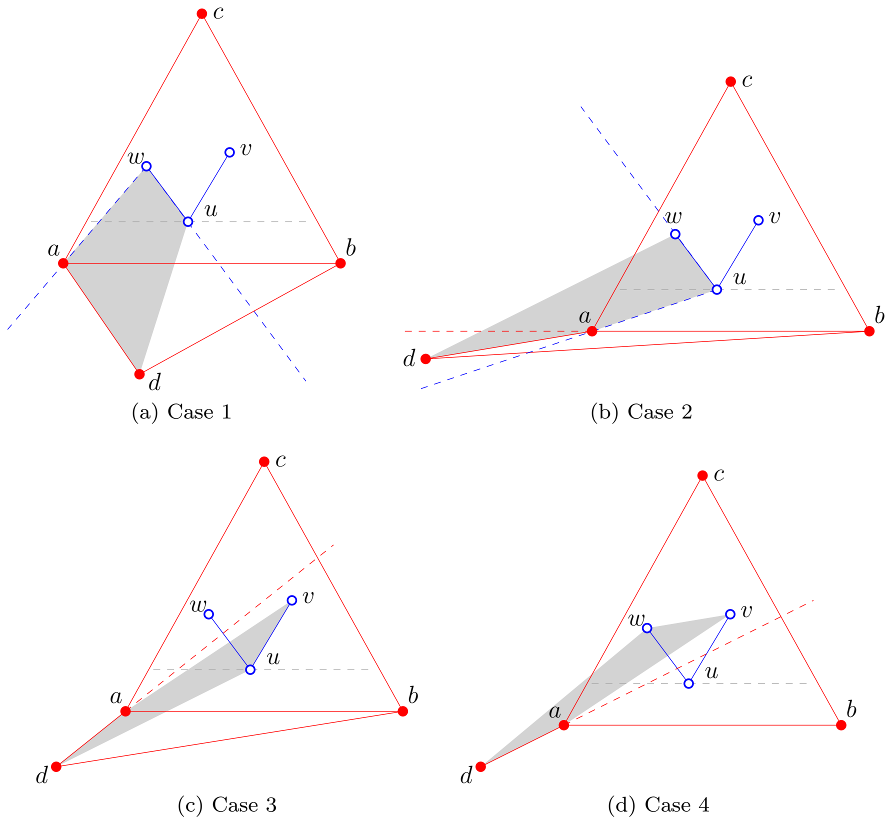
Let $u := f(a, b, c, B)$. Observe that u is a vertex of $CH(B)$. Let $v \in B$ denote the vertex succeeding u in $CH(B)$ in the counter-clockwise order, and $w \in B$ denote the vertex succeeding u in $CH(B)$ in the clockwise order. Both v and w are not below the horizontal line through u by the definition of u. If either $\ell(u, w)$ or $\ell(u, v)$ does not intersect $\overline{ab}$, then there is a balanced convex 4-hole by Lemma 8. Suppose then that both $\ell(u, w)$ and $\ell(u, v)$ intersect $\overline{ab}$. Refer to Figure 8.

We consider the following four cases according to the possible locations of point d, by assuming w.l.o.g. that point d is to the left of $\ell(u, w)$. The other symmetric cases arise when d is to the right of $\ell(u, v)$.

Case 1: $d \in \mathcal{W}(w, a, u)$ (see Figure 8a). The quadrilateral with vertex set $\{a, d, u, w\}$ is a balanced convex 4-hole.

Case 2: $d \in \mathcal{W}(u, a, w)$ (see Figure 8b). The quadrilateral with vertex set $\{d', a, u, w\}$ is a balanced convex 4-hole, where $d' = f(w, a, d, R)$.

Case 3: $d \notin \mathcal{W}(w, a, u) \cup \mathcal{W}(u, a, w)$ and $\ell(a, d) \cap \overline{uv} = \emptyset$ (see Figure 8c). The



(a) Case 1 (b) Case 2

(c) Case 3 (d) Case 4

Figure 8: Proof of Lemma 9.

quadrilateral with vertex set $\{a, d, u, v'\}$ is a balanced convex 4-hole, where $v' = f(a, u, v, B)$.

Case 4: $d \notin \mathcal{W}(w, a, u) \cup \mathcal{W}(u, a, w)$ and $\ell(a, d) \cap \overline{uv} \neq \emptyset$ (see Figure 8d). The quadrilateral with vertex set $\{d', a, v', w\}$ is a balanced convex 4-hole, where $d' = f(a, w, d, R)$ and $v' = f(a, w, v, B)$.

Since any location of d is covered by one of the above cases (or by one of their symmetric ones), there exists a balanced convex 4-hole. The result follows. $\square$

By combining Lemma 7, Lemma 8, and Lemma 9, we obtain the following result that completely characterizes the non-linearly separable bichromatic point sets that have a balanced convex 4-hole.

Theorem 10. *Let $S = R \cup B$ be a bichromatic point set such that R and B are not linearly separable. Then S has a balanced 4-hole if and only if one of the following conditions holds:*

1. *$CH(B) \subset CH(R)$, $|R| = 3$, $|B| \geq 2$, and there is a blue-blue edge $\overline{uv}$ of*

$CH(B)$ *such that one of the open half-planes bounded by* $\ell(u, v)$ *contains exactly 2 red points and no blue point.*

2. $CH(R) \subset CH(B)$, $|B| = 3$, $|R| \geq 2$, *and there is a red-red edge* $\overline{uv}$ *of* $CH(R)$ *such that one of the open half-planes bounded by* $\ell(u, v)$ *contains exactly 2 blue points and no red point.*

3. $CH(B) \subset CH(R)$, $|R| \geq 4$, $|B| \geq 2$,

4. $CH(R) \subset CH(B)$, $|B| \geq 4$, $|R| \geq 2$,

5. *The boundaries of* $CH(B)$ *and* $CH(R)$ *intersect each other.*

3.2 R and B are linearly separable

In the rest of this section, we will assume that R and B are linearly separable. At first glance, one might be tempted to think that if the cardinalities of R and B are large enough, then S always contains balanced convex 4-holes. This certainly happens in the point set of Figure 5, in which R and B are far enough from each other. There are, however, examples of linearly separable bicolored point sets with an arbitrarily large number of points that do not contain any balanced convex 4-hole. For instance, the point set shown in Figure 6d has no balanced convex 4-hole. Observe in this example that if we choose a red-red edge and a blue-blue edge, the convex hull of their vertices is either a triangle or a convex quadrilateral that contains at least one other point in its interior.

Given an edge e of $CH(R)$ and an edge e' of $CH(B)$, we say that e and e' *see each other* if the union of the sets of their vertices defines a balanced convex 4-hole whose interior intersects with neither $CH(R)$ nor $CH(B)$. We assume that there exists a non-horizontal line ℓ such that the elements of R are located to the left of ℓ and the elements of B are located to the right.

Definition 11. *Let* $S = R \cup B$ *be a bicolored point set such that* R *and* B *are linearly separable. Conditions C1 and C2 are defined as follows:*

C1. There exist an edge e *of* $CH(R)$ *and an edge* e' *of* $CH(B)$ *such that* e *and* e' *see each other.*

C2. There exists an edge $\overline{uv}$ *of* $CH(R)$ *and points* $b, z \in B$ *such that* $z \in \Delta uvb$, $R \cap \Delta uvb = \emptyset$, *and* $R \cap \mathcal{W}(b, u, v) \neq \emptyset$; *or this statement holds if we swap* R *and* B.

Lemma 12. *Let* $S = R \cup B$ *be a bicolored point set such that* R *and* B *are linearly separable. If there exist a point* $r \in R$, *a point* $b \in B$, *an edge* e *of* $CH(R)$, *and an edge* e' *of* $CH(B)$, *such that the interiors of* e *and* e' *intersect with the interior of* $\overline{rb}$, *then C1 or C2 holds.*

Proof. Let u and v be the endpoints of e and w and z the endpoints of e'. Assume w.l.o.g. that $\ell(r, b)$ is horizontal, u and w are above $\ell(r, b)$, and then

v and z are below $\ell(r, b)$. If e and e' see each other (see Figure 9a), then *C1* holds. Otherwise, assume w.l.o.g. that z is contained in Δuvw (see Figure 9b). We have $z \in \Delta uvb$ because z lies between the intersections of $\ell(w, z)$ with $\overline{rb}$ and $\overline{uv}$, which both are in the closure of Δuvb. This implies that $R \cap \Delta uvb = \emptyset$ and $r \in \mathcal{W}(b, u, v)$. Then *C2* is satisfied. $\square$

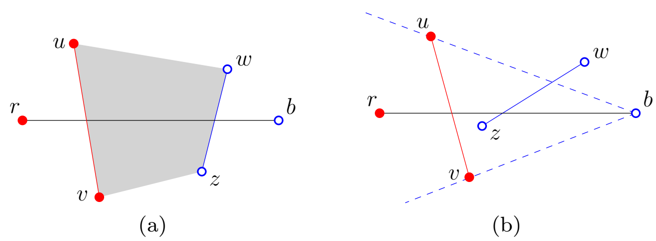


Figure 9: Proof of Lemma 12.

Theorem 13. *A bichromatic point set $S = R \cup B$, such that R and B are linearly separable, has a balanced convex 4-hole if and only if C1 or C2 holds.*

Proof. If condition *C1* holds then S has trivially a balanced convex 4-hole. Then suppose that condition *C2* holds. Let $z' := f(u, v, b, B)$ and observe that $z' \neq b$ since $z \in \Delta uvb$. Let r be any red point in $R \cap \mathcal{W}(b, u, v)$ (see Figure 10a). Observe that we have either $r \in \mathcal{W}(b, u, z')$ or $r \in \mathcal{W}(b, z', v)$. Assume w.l.o.g. the former case. Then the quadrilateral with vertex set $\{r', z', b', u\}$ is a balanced convex 4-hole, where $r' := f(u, z', r, R)$ and $b' := f(u, z', b, B)$.

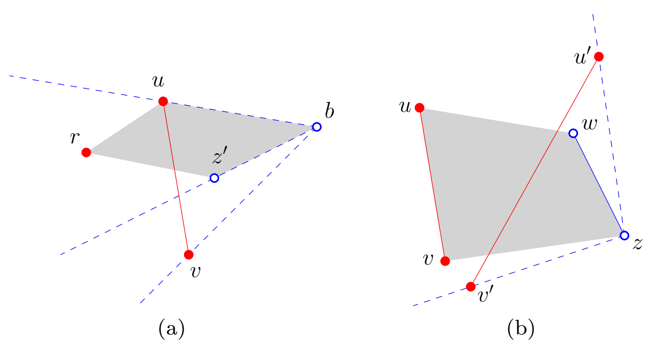


Figure 10: Proof of Theorem 13.

Suppose now that S has a balanced convex 4-hole with vertices u, v, z, w in counter-clockwise order, where $u, v \in R$ and $w, z \in B$. Let e and e' be the edges of $CH(R)$ and $CH(B)$, respectively, that intersect with both $\overline{uw}$ and $\overline{vz}$ (note that e and e' might share vertices with $\overline{uv}$ and $\overline{wz}$, respectively). If we have that $e = \overline{uv}$ and $e' = \overline{wz}$ then e and e' see each other, and thus *C1* holds. Otherwise, if $e \neq \overline{uv}$ and $e' \neq \overline{wz}$ then the interiors of e and e' intersect the interior of the same edge among $\overline{uw}$, $\overline{uz}$, $\overline{vw}$, and $\overline{vz}$. Then, by Lemma 12, we have that *C1* or *C2* holds. Otherwise, there are two cases to consider: (1) $e \neq \overline{uv}$ and $e' = \overline{wz}$; and (2) $e = \overline{uv}$ and $e' \neq \overline{wz}$. Consider case (1), case (2) is analogous. Let $e := \overline{u'v'}$. If e and e' see each other, then *C1* holds. Otherwise (up to symmetry), w belongs to $\Delta u'v'z$ (see Figure 10b). Since $R \cap \Delta u'v'z = \emptyset$ and $u \in \mathcal{W}(z, u', v')$, we have that *C2* is satisfied. $\square$

4 Discussion

A better counting of black edges: In the proof of our lower bounds, we considered the edges colored black, as those being edges of the convex hull of $S = R \cup B$ ($|R| = |B| = n$) that connect a red point with a blue point and are neither an edge nor a diagonal of any balanced 4-hole. Specifically, in the proof of Lemma 3, we gave the simple upper bound $2n$ for the number of black edges, but one can note that this bound can be improved. Nevertheless, any upper bound must be at least $n/2$ since the following bicolored point set has precisely $n/2$ black edges.

Let $n = 4k$ and consider a regular $2k$-gon Q. Put a colored point at each vertex of Q such that the colors of its vertices alternate along its boundary. Orient the edges of Q counter-clockwise. Then for each edge e of Q put in the interior of Q three points of the color of the origin vertex of e such that they are close enough to e and ensure that there is no balanced 4-hole with e as edge. In total we have $8k$ points, consisting of $4k$ red points (i.e. k red points in vertices of Q and $3k$ red points in the interior of Q) and $4k$ blue points. See for example Figure 11, in which $k = 2$. Then, all the $2k = n/2$ edges of Q are black.

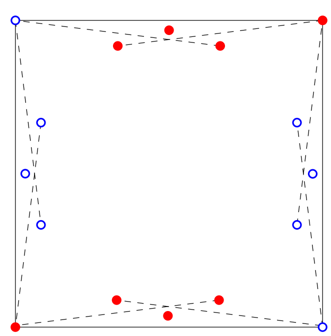


Figure 11: A point set with many black edges.

Generalization of the lower bound for non-balanced point sets: Let $S = R \cup B$ be a red-blue colored point set such that $|R| \neq |B|$. Let $\mathbf{r} := |R|$ and $\mathbf{b} := |B|$. Using arguments similar to the ones used in Section 2, it can be proved that S has at least

$$\mathbf{r}\cdot\mathbf{b} - \mathbf{r}\cdot\min\left\{\left\lfloor\frac{\mathbf{r}-1}{3}\right\rfloor, \mathbf{b}\right\} - \mathbf{b}\cdot\min\left\{\left\lfloor\frac{\mathbf{b}-1}{3}\right\rfloor, \mathbf{r}\right\} - (\mathbf{r}+\mathbf{b})$$

balanced 4-holes. Observe that this bound is positive if and only if $\lfloor\frac{\mathbf{r}-1}{3}\rfloor < \mathbf{b}$ and $\lfloor\frac{\mathbf{b}-1}{3}\rfloor < \mathbf{r}$ (roughly $\mathbf{r} \leq 3\mathbf{b}$ and $\mathbf{b} \leq 3\mathbf{r}$). Therefore, we leave as an open problem to obtain a lower bound for the cases in which the number of points of one color exceeds three times the number of points of the other color.

Existence of convex 4-holes, either balanced or monochromatic: Combining the characterization given by Theorem 10 joint with Theorem 13, we obtain the following result:

Proposition 14. *Let $S = R \cup B$ a bicolored point set in the plane. If $|R|, |B| \geq 4$ then S always has a convex 4-hole either balanced or monochromatic.*

Proof. If R and B are not linearly separable, then S has a balanced convex 4-hole by Theorem 10. Otherwise, consider that R and B are linearly separable. If the convex hull of R contains a red point and the convex hull of B contains a blue point in their interiors, then S has a balanced convex 4-hole by Lemma 12. Otherwise, at least one between R and B is in convex position and then S has a monochromatic convex 4-hole. ☐

Deciding the existence of balanced convex 4-holes: Using the characterization Theorems 10 and 13, arguments similar to those given in Sections 3.1 and 3.2, and well-known algorithmic results of computational geometry, we can decide in $O(n \log n)$ time if a given bicolored point set $S = R \cup B$ ($|R|, |B| \geq 2$) of total n points has a balanced convex 4-hole.

We first compute the convex hulls $CH(R)$ and $CH(B)$ of R and B, respectively. After that, we decide if R and B are linearly separable. If they are not, we can decide in $O(n \log n)$ time whether one of the conditions (1-5) of Theorem 10 holds. Otherwise, if R and B are linearly separable, we proceed with the following steps, each of them in $O(n \log n)$ time. If the decision performed in any of these steps has a positive answer, then a balanced convex 4-hole exists:

1. Decide whether the next two conditions hold: (1) $CH(R)$ contains red points in the interior or $CH(S)$ has at least three red vertices; and (2) $CH(B)$ contains blue points in the interior or $CH(S)$ has at least three blue vertices. If the answer is positive then the conditions of Lemma 12 are met and there thus exists a balanced convex 4-hole in S. Otherwise, if the answer is negative, assume w.l.o.g. that B is in convex position.

2. Decide whether the conditions of Lemma 12 hold for at least one red point r. Fixing a red point r, those conditions can be verified in $O(\log n)$ time as follows: Let $b_0, b_1, \ldots, b_{m-1}$ be all the blue points labelled clockwise along the boundary of $CH(B)$ (subindices are taken modulo m). Let b_i

and b_j be the two blue points such that $r \to b_i$ and $r \to b_j$ are tangent to $CH(B)$, and assume w.l.o.g. that the boundary of $CH(B)$ intersects the interior of the segment $\overline{rb_k}$ for every $k \in [i+1 \dots j-1]$. If r belongs to the interior of $CH(R)$ then it suffices to verify that the range $[i+1 \dots j-1]$ is not empty. Otherwise, if r is a vertex of $CH(R)$, then it suffices to verify whether for some $k \in [i+1 \dots j-1]$ the point b_k is contained in $\mathcal{W}(r, r', r'') \cap \mathcal{W}(r, b_i, b_j)$, where r' and r'' are the vertices preceding and succeeding r, respectively, in the boundary of $CH(R)$. Both b_i and b_j can be found in $O(\log n)$ time, as well the last step can also be performed in $O(\log m) = O(\log n)$ time by applying binary search over the points $b_{i+1}, b_{i+2}, \dots, b_{j-1}$.

3. Decide whether Condition *C1* holds. This can be done in $O(n)$ time by simultaneously traversing the boundaries of $CH(R)$ and $CH(B)$.

4. Decide whether Condition *C2* holds. Using the fact that neither condition *C1* nor the conditions of Lemma 12 hold, we claim that condition *C2* can be decided by assuming that segment $\overline{bz}$ is an edge of $CH(B)$ and that point z is the only blue point in the triangle Δuvb (the condition *C2* with R and B swapped is similar to decide). Namely, let $\overline{uv}$ be an edge of $CH(R)$ and $b, z \in B$ be points such that $z \in \Delta uvb$, $R \cap \Delta uvb = \emptyset$, and $R \cap \mathcal{W}(b, u, v) \neq \emptyset$. Let $z' := f(u, v, b, B) \neq b$, and observe that at least one neighbor of z' in the boundary of $CH(B)$, say b', satisfies $b' \in \Delta uvb \cup \{b\}$ and z' is the only one blue point in $\Delta uvb'$. The fact $R \cap \mathcal{W}(b, u, v) \subseteq R \cap \mathcal{W}(b', u, v)$ implies that we can verify condition *C2* with b' being b and z' being z, where $\overline{b'z'}$ is an edge of $CH(B)$ (see Figure 12a). The claim thus follows. Therefore, there is a linear-size set W of wedges of the form $\mathcal{W}(b, u, v)$ to consider, and we need to check if there is an incidence between any red point and an element of W. Note that the elements of W can be divided into two groups, such that in each group the intersections of the wedges with the interior of $CH(R)$ are pairwise disjoint (see Figure 12b). The wedge $\mathcal{W}(b, u, v)$ goes to the first group when z is the clockwise neighbor of b in the boundary of $CH(B)$, and to the other group otherwise. Then, for each red point r, one can decide in $O(\log n)$ time such an incidence.

Counting balanced 4-holes: Adapting the algorithm of Mitchell et al. [13] for counting convex polygons in planar point sets, we can count the balanced 4-holes of a bicolored point set S of n points in $O(\tau(n))$ time, where $\tau(n)$ is the number of empty triangles of S.

Existence of balanced $2k$-holes in balanced point sets: The arguments used to prove the existence of at least one balanced 4-hole in any point set $S = R \cup B$ with $|R|, |B| \geq 2$ (at the beginning of Section 2) do not directly apply to prove the existence of balanced $2k$-holes in point sets $S = R \cup B$ with $|R|, |B| \geq k$. However, we can prove the following:

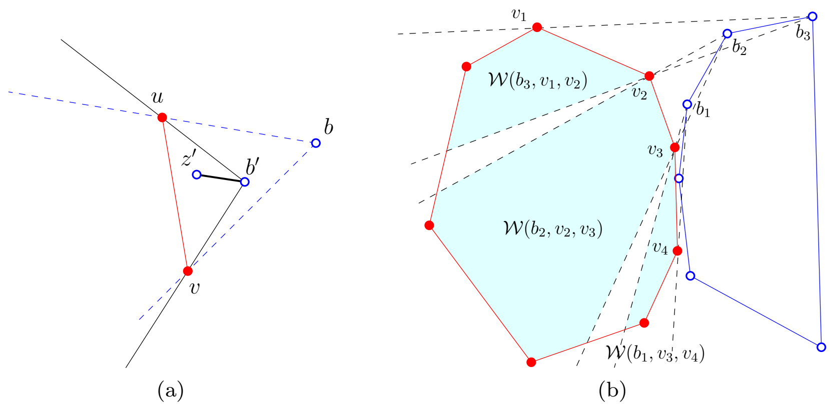


Figure 12: Deciding the existence of a balanced convex 4-hole.

Proposition 15. *For all $n \geq 1$ and $k \in [1..n]$, every point set $S = R \cup B$ with $|R| = |B| = n$ contains a balanced $2k$-hole.*

Proof. If S is in convex position then the result follows. Then, suppose that S is not in convex position. For every point $p \in R$ let $w(p) := 1$, and for every $p \in B$ let $w(p) := -1$. W.l.o.g. let $u \in B$ be a point in the interior of $CH(S)$, and $p_0, p_1, \ldots, p_{2n-2}$ denote the elements of $S \setminus \{u\}$ sorted radially in clockwise order around u. For $i = 0, 1, \ldots, 2n-2$, let $s_i := w(p_i) + w(p_{i+1}) + \ldots + w(p_{i+2k-2})$, where subindices are taken modulo $2n-1$. Notice that all s_i's are odd, and $s_i = 1$ implies that the points $u, p_i, p_{i+1}, \ldots, p_{i+2k-2}$ form a balanced $2k$-hole. We have that $\sum_{i=0}^{2n-2} s_i = (2k-1)\sum_{i=0}^{2n-2} w(p_i) = 2k-1$, which implies (given that $k \in [1..n]$) that not all s_i's can be greater than 1 and that not all s_i's can be smaller than 1. Suppose for the sake of contradiction that none of the s_i's is equal to 1. Then, there exist an $s_j < 1$ and an $s_t > 1$. Since we further have that $s_i - s_{i+1} \in \{-2, 0, 2\}$ for all $i \in [0..2n-2]$, there must exist an element among $s_{j+1}, s_{j+2}, \ldots, s_{t-1}$ which is equal to 1, and the result thus follows. $\square$

Open problems: As mentioned above, we leave as open the problem of obtaining a lower bound for the number of balanced 4-holes in point sets $S = R \cup B$ in which either $|R| > 3|B|$ or $|B| > 3|R|$. Another open problem is to study lower bounds on the number of balanced k-holes, for even $k \geq 6$.

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